Helsinki Center of Economic Research

Discussion Papers

Von Neumann-Morgenstern Stable Sets and the Non-cooperative Solution to the Bargaining Problem

Klaus Kultti
University of Helsinki

and

Hannu Vartiainen
Yrjö Jahnsson Foundation

Discussion Paper No. 114
September 2006

ISSN 1795-0562

HECER – Helsinki Center of Economic Research, P.O. Box 17 (Arkadiankatu 7), FI-00014 University of Helsinki, FINLAND, Tel +358-9-191-28780, Fax +358-9-191-28781,
E-mail info-hecer@helsinki.fi, Internet www.hecer.fi

HECER
Discussion Paper No.114

A Non-cooperative Solution to the Bargaining Problem*

Abstract

We establish a general n-player link between non-cooperative bargaining and the Nash solution. Non-cooperative bargaining is captured in a reduced form through the von Neumann-Morgenstern (1944) stability concept. A stable set always exists. Moreover, if the utility set has a smooth surface, then any stable set converges to the Nash bargaining solution. Finally, the equivalence of stationary equilibria of the unanimity bargaining game and the stable set solution is demonstrated.

JEL Classification: C71, C78

Keywords: von Neumann-Morgenstern stable set, Nash bargaining solution, non-cooperative bargaining.

Klaus Kultti

Department of Economics,
University of Helsinki
P.O. Box 17 (Arkadiankatu 7)
FI-00014
FINLAND

e-mail: klaus.kultti@helsinki.fi

Hannu Vartiainen

Yrjö Jahnsson Foundation
Ludviginkatu 3-5
FIN00130 Helsinki
FINLAND

e-mail: hannu.vartiainen@yjs.fi

* We thank Hannu Salonen for useful comments.

1 Introduction

An $n-$player bargaining problem is defined by an $n-$dimensional compact, comprehensive, and convex utility possibility set U. Nash's (1950) solution is without doubt the most commonly accepted cooperative solution to the problem.[1] But this constitutes only part of the story; a more complete theory would explain why the Nash solution emerges also in the *noncooperative* framework.

The aim of this paper is to establish a general link between the noncooperative approach and the Nash solution. Noncooperative bargaining is captured in a reduced form through the von Neumann-Morgenstern (1944) stability concept.[2] We show that a *stable set* always exists (see below for a definition). Moreover, if the utility set has a *smooth* surface, then any stable set converges to the Nash bargaining solution. Finally we demonstrate the equivalence of *stationary* equilibria of a unanimity bargaining game and the stable set solution. As no assumptions are made as regards to the underlying physical environment,[3] the model proposes a general noncooperative foundation for the Nash bargaining solution.

The relation between the Nash solution and noncooperative bargaining is of course not new. Binmore *et al.* (1986) show that the unique equilibrium outcome of the two-player Rubinstein (1982) alternating offers bargaining game converges to the Nash solution when the time difference between offers becomes small. It is well known that the same is true also for the *stationary* equilibria in the $n-$player *cake sharing* context (see e.g. Osborne and Rubinstein, 1990, and Chatterjee and Sabourian, 2000).[4] However, the assumption of private consumption is indispensable; *resource monotonicity* of the solution (more cake implies bigger equilibrium shares for all players), which is implied by this assumption, drives the uniqueness and convergence results.

Much of the literature has focused on the issue of how to live without the stationarity assumption, and less attention has been paid to the sensitivity of the outcome to the underlying structure. Nevertheless, relaxing the private consumption structure has non-trivial consequences. There are examples

[1] For an authorative discussion on n-player bargaining theory, see Thomson and Lensberg (1989).

[2] For a review, see Owen (1989).

[3] Except that it induces a smooth bargaining problem.

[4] In an important paper, Krishna and Serrano (1996) construct an $n-$player cake sharing game whose unique subgame perfect equilibrium outcome converges to the Nash solution.

of compact, convex, and comprehensive utility possibility sets that induce peculiar stationary equilibria. In particular, there may be many stationary equilibrium, and no such equilibrium need to converge to the Nash solution.[5] Even the existence of a stationary equilibrium has been unclear. We show that the existence is never an issue, and the convergence is guaranteed with the smooth utility frontier -assumption.

The Solution The stable set solution is derived by assuming the following. Any player may impose an objection to a division of utilities by demanding a new division. It takes one period before any such demand may materialize. A dominance relation over the divisions of utilities is imposed: a division u is dominated by a division v if and and only if the discounted value of v exceeds the current value of u for *some* player.

With a short enough period there are no undominated divisions. We focus on a subset of all divisions, the stable set, defined as follows: any element of the stable set can only be dominated by an element outside the set, and any element outside the stable set is dominated by some element of the set. We show that the $i-$maximal points of a stable set correspond to stationary equilibrium outcomes of a unanimity bargaining game. Thus the stable set reflects stationary behavior in a reduced form.

Our notion of a stable set is closely related to the "Nash-like solution" of Thomson and Lensberg (1989). The connection of the two concepts is discussed in the final section.

2 The set up

There is a set $N = \{1, ..., n\}$ of players and a compact, convex and comprehensive utility possibility set $U \subset \mathbb{R}^n_+$.[6] [7] The vector of utilities is denoted by $u = (u_1, ..., u_n)$, or $u = (u_i, u_{-i})$. For any $v \in U$, let $D(v)$ be the points that Pareto dominate $v \in U$:

$$D(v) := \{u \in U : u \geq v\}. \tag{1}$$

For any $v \in U$, $D(v)$ is a compact and $v-$comprehensive set. Pareto-optimal outcomes P are then defined by $P := \{u \in U : D(u) = \{u\}\}$.

[5] Non-convergence is possible in non-smooth cases. Examples can be based on Thomson and Lensberg (1989), pp. 120-4.

[6] Vector notation: $x \geqq y$ if $x_i \geq y_i$ for all i, $x \geq y$ iff $x \geqq y$ and not $x_i = y_i$ for all i, and $x > y$ iff $x_i > y_i$ for all i.

[7] $X \subset \mathbb{R}^k$ is $d-$comprehensive if $x \in X$ and $x \geq y \geq d$ imply $y \in X$. If $d = 0$, then X is comprehensive.

Bargaining takes place through objections against a potential division of utilities. An objection is a specification for a new division. However, there is a one-period delay before an objection may become effective. Delay is costly: The present value of player i's next period utility u_i is $u_i\delta_i^{\Delta}$, where $0 < \delta_i < 1$ is the discount factor and $\Delta \geq 0$ is the length of the period.

An abstract stable set is defined with respect to a domain alternatives and a dominance relation on this set. We let the domain be U. Dominance relation $\succ$ is defined as follows: $u \succ v$ iff $u_i\delta_i^{\Delta} > v_i$, for some $i \in N$, for $u, v \in U$. Set $G \subset U$ is *stable* if

- (External stability) $u \notin G$ implies there is $v \in G$ s.t. $v \succ u$,
- (Internal stability) $u \in G$ implies there is *no* $v \in G$ s.t. $v \succ u$.

3 Characterization and Existence

If $\Delta = 0$ then a stable set has a simple structure: G is a stable set if and only if $G = \{u\}$ for any $u \in P$. To see the only if-part, assume that G is a stable set. By internal stability, G contains a single element, say u. But by external stability, $u \succ v$ for all $v \neq u$. Thus $u \in P$.

From now on, assume that $\Delta > 0$. Without loss of generality, let $\Delta = 1$. Take $u = (u_1, ..., u_n)$, and call $(\delta_i^{-1}u_i, u_{-i})$ the δ_i-extension of $u \in U$.[8] Denote by $\underline{u}$ a typical point whose all δ_i-extensions lie on the Pareto-frontier is :

$$\underline{u} \in \{u \in U : (\delta_i^{-1}u_i, u_{-i}) \in P, \text{ for all } i \in N\}. \tag{2}$$

Occasionally, such allocation is called a "minimal point".

For any nonempty set $X \subset U$, define the supremum of i's feasible payoffs in X by

$$m_i(X) = \sup\{u_i : u \in X\}.$$

Theorem 1 *A set $G \subset U$ is stable if and only if $G = D(\underline{u})$.*

Proof. "If": Assume that $G = D(\underline{u})$. By construction, $u_i \geq \underline{u}_i = m_i(G)\delta_i \geq v_i\delta_i$, for all $i \in N$, for all $u, v \in D(\underline{u})$. Thus, internal stability is met. Take $u \notin D(\underline{u})$. Then there is a player $i \in N$ such that $\underline{u}_i > u_i$. This implies that also $\underline{u}_i = m_i(G)\delta_i > u_i$. Since $m_i(G) \in \{u_i : u \in D(\underline{u})\}$, the external stability is met.

[8] The concepts are taken from Thomson - Lensberg (1989), Ch 8.

"Only if": Suppose G is a stable set. Then, by external stability,

$$v \notin \bigcap_{i \in N} \{u \in U : m_i(G)\delta_i \leq u_i\} \text{ implies } v \notin G. \tag{3}$$

By internal stability,

$$v \in \bigcap_{i \in N} \{u \in U : m_i(G)\delta_i \leq u_i\} \text{ implies } v \in G. \tag{4}$$

Thus

$$\bigcap_{i \in N} \{u \in U : m_i(G)\delta_i \leq u_i\} = G.$$

By (4), G is compact. Since U is a comprehensive set, there is $\underline{u} = (\underline{u}_1, ..., \underline{u}_n) \in U$ such that $m_i(G)\delta_i = \underline{u}_i$ for all $i \in N$. By construction, $G = \{u : u \geq \underline{u}\}$. Then $G = D(\underline{u})$ for D meeting (1), and $\underline{u}$ meeting (2), as required. ■

Thus, now we know that any stable set has a particular structure. A stable set is characterized by a minimal point $\underline{u} = (\underline{u}_1, ..., \underline{u}_n)$: points in U above $\underline{u}$ constitute a stable set. Moreover, a stable set is convex, and contains n "maximal points" $u^1, ..., u^n$ that induce the highest possible payoff in the stable set for each $1, ..., n$. Given the minimal point $\underline{u}$, player i's maximal point satisfies $u^i = (\delta_i^{-1}\underline{u}_i, \underline{u}_{-i})$. Also, if u^i is an $i-$maximal point of a stable set G, then $m_i(G) = u_i^i$.

The characterization leaves open the question of existence and uniqueness. Indeed, it is clear in general there may be many stable sets, and in some times it fails to exist. We prove that in our domain the existence is guaranteed.

Recall that for any $u = (u_i, u_{-i})$, we denote the i's maximal payoff fixing the other players' payoffs at u_{-i} by

$$m_i(D(u_i, u_{-i})) = \max\{u_i' : (u_i', u_{-i}) \in U\}.$$

If $u \in P$, then $m_i(D(u)) = u_i$.

Theorem 2 *A stable set exists.*

Proof. Define function $g_i : U \to \mathbb{R}_+$

$$g_i(u) := \delta_i m_i(D(u)), \text{ for all } (u_i, u_{-i}) \in U, \text{ for all } i \in N. \tag{5}$$

By convexity of U, g_i is a continuous function. Let $g(\cdot) := (g_1(\cdot), ..., g_n(\cdot))$, and define function $\bar{x} : U \to \mathbb{R}_+$ such that

$$\bar{x}(u) := \max\{x \in \mathbb{R} : xg(u) \in U\}, \text{ for all } u \in U.$$

By compactness of U, $\bar{x}$ is well defined. Construct function $\hat{g}_i : U \to \mathbb{R}_+$

$$\hat{g}_i(u) := g_i(u) \min\{\bar{x}(u), 1\}, \text{ for all } u \in U.$$

If $\min\{\bar{x}(u), 1\} = 1$, then $\hat{g}(u) \in U$, and if $\min\{\bar{x}(u), 1\} = \bar{x}(u)$, then $\hat{g}(u) = \bar{x}(u)g(u) \in U$. Thus,

$$\hat{g}(u) = (\hat{g}_1(u), ..., \hat{g}_n(u)) : U \to U.$$

By convexity of U, function $\bar{x}$ is continuous. Thus $\hat{g} : U \to \mathbb{R}^n_+$ is a continuous function. By Brouwer's Theorem, there is a vector $\underline{u} \in U$ such that

$$\hat{g}(\underline{u}) = \underline{u}. \tag{6}$$

If also

$$g(\underline{u}) \in U, \tag{7}$$

then, $g(\underline{u}) = \underline{u}$. This implies that $\underline{u}$ satisfies condition (2), and that $D(\underline{u})$ is a stable set. Thus condition (7) needs to be checked.

Suppose (7) does not hold. Then

$$\bar{x}(\underline{u}) < 1. \tag{8}$$

By (6) and (8),

$$\underline{u} = g(\underline{u})\bar{x}(\underline{u}) \in P. \tag{9}$$

This implies that $m_i(D(\underline{u})) = \underline{u}_i$, for all $i \in N$. By (5) and convexity of U we have

$$\begin{aligned} g(\underline{u}) &= (\delta_1 m_1(D(\underline{u})), ..., \delta_n m_n(D(\underline{u}))) \\ &= (\delta_1 \underline{u}_1, ..., \delta_n \underline{u}_n) \\ &= \delta \underline{u} \\ &\in U, \end{aligned}$$

a contradiction. Thus $g(\underline{u}) \in U$, as required. ■

The Existence Theorem is based on the convexity of U. Kultti and Vartiainen (2003) show that in the cake division problem, which imposes a degree of independency on players' payoffs, the stable set is unique.[9] The method of proof is to first show that in any two-player bargaining problem

[9] The result relies on the Fisburn and Rubinstein (1980) specification of time consistent preferences.

the solution is unique. Then we show that such a solution is monotonic w.r.t. to the size of the cake. This in turn implies that if there are two distinct solutions for the general problem, then the minimal point of one of them Pareto dominates the minimal point of the other, which cannot be the case. However, in the current, unrestricted case, uniqueness cannot be ensured. For an example of such case, see the next section.

4 Relationship with the Nash solution

We now argue that there is a particular relation between a the stable set and the Nash bargaining solution. Denote by G_Δ *a* stable set when the length of the period is Δ (there may be many stable sets). We are mainly interested in the limit behavior of G_Δ when Δ becomes small.

First, introduce a vector of weights $\alpha = (\alpha_1, ..., \alpha_n)$ where

$$\alpha_i = \frac{-1}{\ln \delta_i}, \quad \text{for all } i \in N.$$

Denote the $\alpha -$ *weighted Nash solution* by

$$u^\alpha := \arg\max_{u \in U} \prod_{i \in N} u_i^{\alpha_i}. \tag{10}$$

Also denote by

$$H(u) := \left\{ (v_1, ..., v_n) \in \mathbb{R}^n : \prod v_i^{\alpha_i} = \prod u_i^{\alpha_i} \right\},$$

the $\alpha-$weighted hyperbola that contains u.

For any $\Delta > 0$, denote by $\underline{u}(\Delta)$ the minimal point and by $u^1(\Delta), ..., u^n(\Delta)$ the maximal points of the stable set G_Δ, i.e., for all i,

$$u^i(\Delta) = (\delta_i^{-\Delta} \underline{u}_i(\Delta), \underline{u}_{-i}(\Delta)).$$

Lemma 3 *$u^1(\Delta), ..., u^n(\Delta)$ lie on the same hyperbola.*

Proof. Recall that

$$\begin{aligned} \prod_i u_i^j(\Delta)^{\alpha_i} &= \prod_i \delta_i^{-\Delta\alpha_i} \underline{u}_i(\Delta)^{\alpha_i} \\ &= \delta_i^{-\Delta\alpha_i} \prod_i \underline{u}_i(\Delta)^{\alpha_i} \\ &= e^\Delta \prod_i \underline{u}_i(\Delta)^{\alpha_i}, \end{aligned}$$

which is independent of $j \in N$. Thus $u^j(\Delta) \in H(u^i(\Delta))$, for all i, j. ■

First we establish that in the two-player case the Nash solution always belongs to G_Δ.

Theorem 4 *Let $n = 2$. Then $u^\alpha \in G_\Delta$, for all $\Delta > 0$.*

Proof. By Lemma (3), $u^j(\Delta) \in H(u^i(\Delta))$, and hence the intersection of G_Δ and the Pareto frontier is the arc between the intersection of a hyperbola and the Pareto frontier. Since different hyperbolas are nested, and the Pareto frontier concave, the intersection of the highest hyperbola and the Pareto frontier must belong to G_Δ. Hence $u^\alpha \in G_\Delta$. ■

An immediate corollary of the previous theorem is that since the stable set necessarily becomes "small" when Δ tends to zero, and since the Nash solution always belongs to the stable set, it follows that the stable set actually shrinks to the Nash solution as Δ tends to zero.

Unfortunately, it need not be the case that $u^\alpha \in G_\Delta$ when $n > 2$; $G_\Delta \cap G_{\Delta'}$ may be empty, for some Δ, Δ' in such case. However, we are able to establish a weaker convergence result.

We say that a sequence $\{G_\Delta\}$ of stable sets converges to $\{u\}$ in the Hausdorff metric as Δ tends to zero if for any open ball with radius r around $u \in U$, denoted by $B^r(u)$, there is $\Delta_r > 0$ such that $G_\Delta \subset B^r(u)$, for all $\Delta \in (0, \Delta_r)$.

Theorem 5 *Let P be smooth. Then* any *sequence of stable sets converges to $\{u^\alpha\}$ as Δ tends to* 0.

The proof can be summarized as follows. Consider the three (n) player case (see Fig. 1). Think of the surface P of U as a chart of 1-dimensional curves ($n-2$ -dimensional manifolds), each reflecting an intersection of P and a hyperbola. As Δ becomes small, a (sub)sequence of stable sets shrinks to a point u^* on P. If u^* is distinct from u^α, then, since P is smooth, the envisioned chart over P is locally homeomorphic to an open disk that is permeated by a collection of line segments, each corresponding to a hyperbola. Any neighborhood of u^* also contains the maximal points of the stable set for small enough Δ. Under any Δ, the maximal points lie on the same hyperbola, and they span a 2 -dimensional simplex T. Thus, it follows that T becomes embedded into a line segment as Δ tends 0, which leads to a contradiction.

[FIGURE 1 AROUND HERE]

Proof. Let $\{G_\Delta\}_{\Delta>0}$ be a collection of stable sets. For any Δ, let $\underline{u}(\Delta)$ be the minimal point of G_Δ, and $u^1(\Delta), ..., u^n(\Delta) \in P$ the corresponding maximal points. By (3), there is a unique $n-1$ -dimensional hyperplane $L(\Delta)$ that contains $u^1(\Delta), ..., u^n(\Delta)$. Since $(\delta_i^{-\Delta} - 1)\underline{u}_i(\Delta)$ tends to zero as Δ becomes small, there is a subsequence $\{\Delta\}$ converging to zero such that, for some $u^* \in P$, and some hyperplane L^*,[10]

$$\begin{aligned} u^i(\Delta) &\to u^*, \text{ for all } i, & (11)\\ L(\Delta) &\to L^*. & (12)\end{aligned}$$

Denote the hyperplane that supports $H(u)$ at $u \in U$ by $L^H(u)$. Since P is smooth, (11) implies that

$$L^H(u^i(\Delta)) \to L^*, \text{ for all } i. \tag{13}$$

Denote by $V(u)$ the $n-2$ dimensional hyperplane that supports $H(u)$ at $u \in L(\Delta)$ in the subspace $L(\Delta)$. Then

$$V(u^i(\Delta)) = L^H(u^i(\Delta)) \cap L(\Delta), \text{ for all } i.$$

Suppose, to the contrary of the theorem, that $u^* \neq u^\alpha$. Since $V(u^i(\Delta))$ and $V(u^j(\Delta))$ support the same hyperbola in the same subspace and, by (12) and (13), they approach the same limit, we have

$$\frac{\min_v \left\{ \left\| u^j(\Delta) - v \right\| : v \in V(u^i(\Delta)) \right\}}{\| u^j(\Delta) - u^i(\Delta) \|} \to 0, \text{ for all } j \neq i, \tag{14}$$

as depicted in Fig. 2.

[FIGURE 2 AROUND HERE]

[10] Since, for any Δ, there are $p \in [0,1]^n$ and $u^i(\Delta) \in U$ such that $L(\Delta) = u^i(\Delta) + \{v \in \mathbb{R}^n : v \cdot p = 0\}$, the set of parameters defining $\{L(\Delta)\}_{\Delta>0}$ is bounded.

For any $c \in \mathbb{R}_+^n$, denote by A_c the linear transformation matrix

$$A_c = \begin{bmatrix} c_1 & 0 & \cdots & 0 \\ 0 & c_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & c_n \end{bmatrix}.$$

Abusing the notation, $A_c x = (c_1 x_1, ..., c_n x_n)$ and $A_c X = \{A_c x' : x' \in X\}$, for any $x \in X \subset \mathbb{R}_+^n$. Given Δ, choose $c(\Delta) = (c_1(\Delta), ..., c_n(\Delta))$ such that

$$c_i(\Delta) = (\delta_i^{-\Delta} - 1)\underline{u}_i(\Delta), \text{ for all } i \in N. \tag{15}$$

Denote the $n-1$ -dimensional standard simplex by $T = \left\{x \in \mathbb{R}_+^n : \sum x_i = 1\right\}$. Now[11]

$$\mathrm{co}\{u^1(\Delta), ..., u^n(\Delta)\} = \underline{u}(\Delta) + A_{c(\Delta)}T, \text{ for all } \Delta. \tag{16}$$

Fix some player i. Define

$$j(\Delta) = \arg\max_j \left[\min_v \left\{\left\|u^j(\Delta) - v\right\| : v \in V(u^i(\Delta))\right\}\right].$$

Take any $\varepsilon > 0$. By (14), there is Δ_ε such that, for all $\Delta \in (0, \Delta_\varepsilon)$,

$$\frac{\min_v \left\{\left\|u^{j(\Delta)}(\Delta) - v\right\| : v \in V(u^i(\Delta))\right\}}{\left\|u^{j(\Delta)}(\Delta) - u^i(\Delta)\right\|} < \varepsilon. \tag{17}$$

By (16) and (17), for all $\Delta \in (0, \Delta_\varepsilon)$,

$$\underline{u}(\Delta) + A_{c(\Delta)}T \quad \subset \quad \left\{u : \|u - v\| < \varepsilon \left\|u^{j(\Delta)}(\Delta) - u^i(\Delta)\right\|, \text{ for } v \in V(u^i(\Delta))\right\}. \tag{18}$$

Since, by (15),

$$\left\|A_{c(\Delta)}^{-1}u^{j(\Delta)}(\Delta) - A_{c(\Delta)}^{-1}u^i(\Delta)\right\| = \sqrt{2},$$

condition (18) reduces to

$$T \subset \left\{u : \|u - v\| < \varepsilon\sqrt{2}, \text{ for } v \in V(A_{c(\Delta)}^{-1}[u^i(\Delta) - \underline{u}(\Delta)])\right\}, \text{ for all } \Delta \in (0, \Delta_\varepsilon).$$

I.e., T is contained by the $\varepsilon\sqrt{2}-$neighborhood of the hyperplane $V(A_{c(\Delta)}^{-1}[u^i(\Delta) - \underline{u}(\Delta)])$, defined with respect to the utility possibility set $A_{c(\Delta)}^{-1}[U - \underline{u}(\Delta)]$. But since $\varepsilon > 0$ is arbitrarily small, this means that the $n-1$ -dimensional simplex T is contained by an $n-2$ -dimensional hyperplane, a contradiction. ■

[11] co$X$ is a convex hull of $X \subset \mathbb{R}^n$.

4.1 Counter examples

Convergence need not hold if P is not smooth. For an example, consider the three player "pyramid" case $U = \{x \in \mathbb{R}^3_+ : x_1 + 2\max_{i=2,3}\{x_i\} \leq 2\}$. Assume equal discount factors δ. The minimal point $\underline{u}$ of the *unique* stable set is defined by

$$(\underline{u}_1, \underline{u}_2, \underline{u}_3) = \left(\frac{2\delta^\Delta}{1+\delta^\Delta}, \frac{\delta^\Delta}{1+\delta^\Delta}, \frac{\delta^\Delta}{1+\delta^\Delta}\right).$$

This converges to $(1, 1/2, 1/2)$ as Δ tends to 0. However, the Nash solution for the problem is $(2/3, 2/3, 2/3)$ (see Fig. 3). Hence smoothness is crucial for the convergence result. Since a non-smooth utility space can be approximated by smooth ones, there is an important discontuity in convergence of stable sets.

[FIGURE 3 AROUND HERE]

Now we use U to construct an example of a case qirh multiple stable sets. First, identify a triangular problem $V = \{x \in \mathbb{R}^3_+ : x_1 + x_2 + x_3 \leq 2\}$ (see Fig. 4). The minimal point $\underline{v}$ of the unique stable set related to this is

$$(\underline{v}_1, \underline{v}_2, \underline{v}_3) = \left(\frac{2\delta}{1+2\delta}, \frac{2\delta}{1+2\delta}, \frac{2\delta}{1+2\delta}\right).$$

Now perturb U by substracting ε from the 1's maximal payoff and adding ε to the maximal payoffs of 2 and 3. Denote the resulting utility space by,

$$U^\varepsilon = \left\{x \in \mathbb{R}^3_+ : x_1 + \left(\frac{2-\varepsilon}{1+\varepsilon}\right)\max_{i=2,3}\{x_i - \varepsilon\} \leq 2 - \varepsilon\right\}.$$

The convergence point u^* of the unique stable set of this problem is

$$u^* = \left(1 - \frac{\varepsilon}{2}, \frac{1}{2} + \frac{\varepsilon}{2}, \frac{1}{2} + \frac{\varepsilon}{2}\right).$$

Now, let W be the intersection of U^ε and V (see Fig 5). The convergence point u^* of U^ε belongs to V and hence it belongs to W. In the neighborhood of u^*, W coincides with U^ε. The Nash solution of V belongs to U^ε and hence it belongs to W. In the neighborhood of this point, W coincides with U^ε. Thus, for small enough Δ, both neighborhoods contain a stable set implying that for such δ^Δ there are two stable sets (this property is not related to the non-smoothness of the surfaces). For closely related analysis, see Thomson and Lensberg, 1989, Ch. 8.2.

[FIGURE 4 AND 5 AROUND HERE]

5 Relation to a Bargaining Game

In this section, we assume that U is strictly convex. We study the standard unanimity bargaining game Γ. At any stage $t \in \{0, 1, ...\}$,

- Player $i(t)$ makes an offer $v \in U$. Players $j \neq i(t)$ accept or reject the offer in the ascending order of their index.
- If all $j \neq i(t)$ accept, then v is implemented. If j is the first who rejects, then j becomes $i(t+1)$.
- $i(0) = 1$.

We concentrate on the stationary equilibria of the game, where:

1. Each $i \in N$ makes the same proposal $v(i)$ whenever he serves as the proposer.
2. Each i's acceptance decision in period t depends only on v_i that is offered to him in that period.

Next we show that any stationary equilibrium constitutes a maximal point of a stable set (let $\Delta = 1$).

Proposition 6 *v is a stationary equilibrium outcome of Γ if and only if it is a $1-$maximal point of a stable set.*

Proof. Only if: In any equilibrium, $i(t)$'s offer $v(i(t)) = (v_j(i(t)))_{j \in N}$ is accepted at stage $t \in \{0, 1...\}$. In stationary equilibrium the time index t can be relaxed from $v(i(t))$. An offer v_j by player i is accepted by all $j \neq i$ if

$$v_j \geq \delta v_j(j), \text{ for all } j \neq i. \tag{19}$$

Player i's equilibrium offer $v(i)$ maximizes his payoff with respect to constraint (19). By the comprehensiveness and strict convexity of U, all constraints in (19) must bind, and $v(i) \in P$. I.e.

$$v_j(i) = \delta v_j(j), \text{ for all } j \neq i.$$

Thus, by Theorem 1, $D(\delta v_i(i), v_{-i}(i))$ constitutes a stable set and $v(i)$ its $i-$maximal point. Choosing $t = 0$ and $i(0) = 1$, gives the result.

If: Choose a stable set. Let v^j be its $j-$maximal point, for any $j \in N$. Then, by Theorem 1,

$$v_j^i = \delta v_j^j, \text{ for all } j \neq i.$$

Construct the following stationary strategy: Player i always offers v^i and accept anything above δv_i^i. Player i's offer v is accepted by all $j \neq i$ only if

$$v_j \geq \delta v_j^j, \text{ for all } j \neq i. \tag{20}$$

Since v^i maximizes i's payoff given (20), a deviation by i cannot be profitable. Given this, a deviation in acceptance cannot increase i's payoff either since in the next period he would get payoff at most v_i^i. ■

Since a stable set converges to the Nash bargaining solution u^α as the time interval becomes small, it follows by Proposition 6 that also *all* stationary equilibria associated to Γ converge to u^α.

Corollary 7 *A stationary equilibrium of* Γ *exists and converges to* u^α *as* Δ *tends to zero.*

6 Discussion

The current paper is closely related to Thomson (1988), Lensberg (1988), and Thomson and Lensberg (1989), Ch. 7 and 8, where the axiom of multilateral (population) stability is imposed on solutions, and the notion of "Nash-like" solution is developed. It can be shown that the minimal point of the stable set is a particularly parametrized Nash-like solution.

Thomson and Lensberg discuss the convergence properties of the Nash-like solution. They point out that in a smooth problem the Nash-like solution converges to the Nash solution. Their reasoning is based on the fact that if in a smooth problem all two-player components of a Pareto-optimal point constitute bilateral Nash-solutions in their respective two-player utility projections, then the Pareto-optimal point also constitutes a Nash solution of the whole problem.[12]

In addition, Thomson and Lensberg note that the existence of a Nash-like solution can be obtained by using Varian's (1981) result on the existence of a fixed points in continuous vector fields. To the contrary, our proof appeals to the familiar Brouwer Fixed Point Theorem.

Rubinsten, Safra, and Thomson (1992) characterize the Nash solution in a model whose motivation bears similarity to our framework. They use a system of objections and counterobjections to define the Nash solution. There are important differences to our approach. First, the Rubinstein *et al.* model is relies on an asymmetry between objections and counterobjections. Hence their solution does not conceptually relate to the stable set (except in terms of convergence).

To see this, let us formulate the Rubinstein *et al.* solution by saying that an outcome u is $p-$*unstabilized* by u' if $pu'_i > u_i$ and $pu_j \leq u'_j$ for some $i \neq j$. Rubinstein *et al.* show that the unique intersection of the not $p-$unstabilized outcomes, $p \in (0,1)$, is the Nash solution. However, the set of $p-$stable allocations does not conceptually relate to the stable set under $\delta^{\Delta} = p$. For one thing, the internal stability need not hold for $p-$stable outcomes as such outcome may well be an objection against another $p-$stable allocation (if two outcomes are each another's objections then they are also counterobjections). For another thing, nothing guarantees the external stability since a $p-$unstable outcome need not be objected via a $p-$stable one.

Second, Rubinstein *et al.* analyse the two players case, and it is not clear how to extend it to a multi-player scenario. A straightforward extension would allow only bilateral objections/counterobjections.[13] Along the argument made in this paper, that would lead to a the Nash solution in the class of *smooth* problems.

[12] Given this, the result follows since in smooth problems the bilateral coordinates of the minimal point are continuous in Δ.

[13] Along the two-player projections, and keeping the other players' payoffs fixed.

References

[1] Binmore K., A. Rubinstein and A. Wolinsky 1986, The Nash bargaining solution in economic modelling, *Rand Journal of Economics 17*, 176-188.

[2] Chatterjee, K. and H. Sabourian 2000, Multiperson Bargaining and Strategic Complexity, *Econometrica* **68**, 1491-1509

[3] Fishburn, P. and Rubinstein, A. 1982, Time Preference, *International Economic Review 23*, 677-95.

[4] Greenberg J. 1990, *The Theory of Social Situations.* Cambridge University Press, Cambridge.

[5] Krishna V. and R. Serrano 1996, Multilateral bargaining, *Review of Economic Studies 63*, 61-80.

[6] Kultti, K. and H. Vartiainen 2003, Von Neumann-Morgenstern solution to the cake division problem, *manuscript.*

[7] Lensberg, T. 1988, Stability and the Nash Solution, *Journal of Economic Theory 54*, 101-16.

[8] Lensberg, T. and W. Thomson 1988, Characterizing the Nash Solution without Pareto-optimality, *Social Choice and Welfare 5,* 247-59.

[9] Nash J. 1950, The bargaining problem, *Econometrica 18*, 155-162.

[10] Owen G. 1995, *Game theory*, New York: Academic Press.

[11] Osborne, M. and Rubinstein A. 1990, *Bargaining and Markets*, Academic Press.

[12] Rubinstein A. 1982, Perfect equilibrium in a bargaining model, *Econometrica 50*, 97-109.

[13] Rubinstein A., Safra, Z. and W. Thomson 1992, On the Interpretation of the Nash Bargaining Solution and its Extension to Non-Expected Utility Preferences, *Econometrica 60*, 1171-1186.

[14] Thomson W. and T. Lensberg 1989, *Axiomatic theory of bargaining with a variable number of agents.* Cambridge University Press, Cambridge.

[15] von Neumann J. and O. Morgenstern 1944, *Theory of Games and Economic Behavior*. New York: John Wiley and Sons.

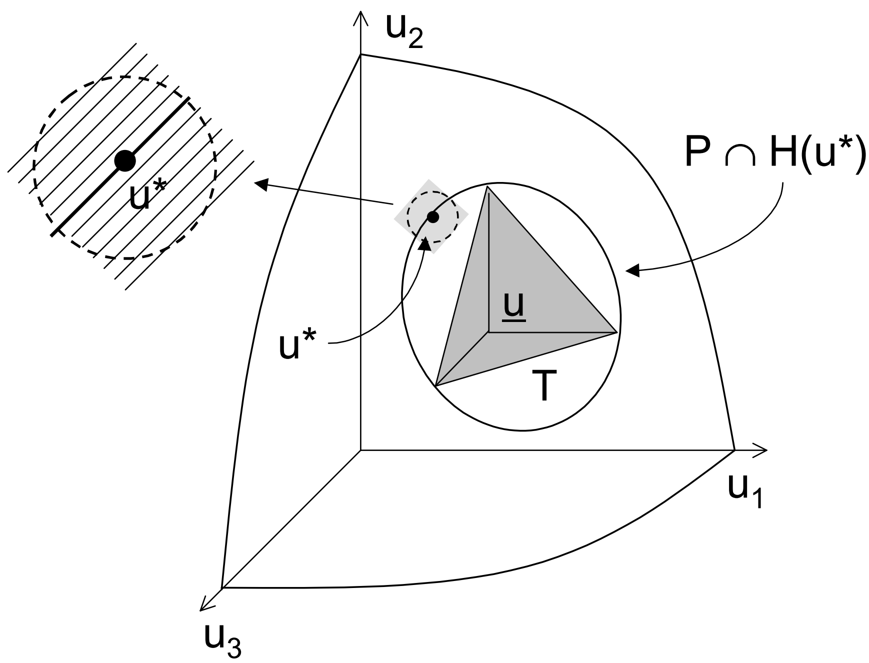

Figure 1

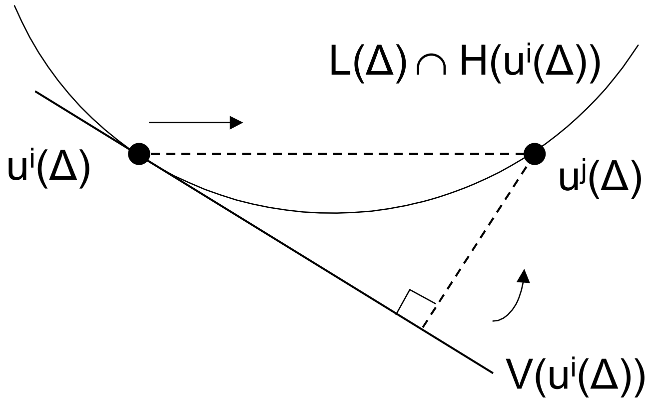

Figure 2

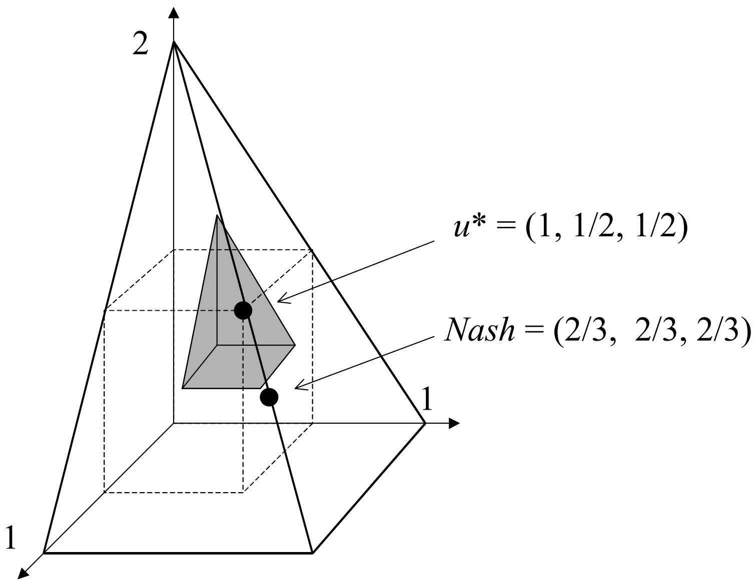

Figure 3

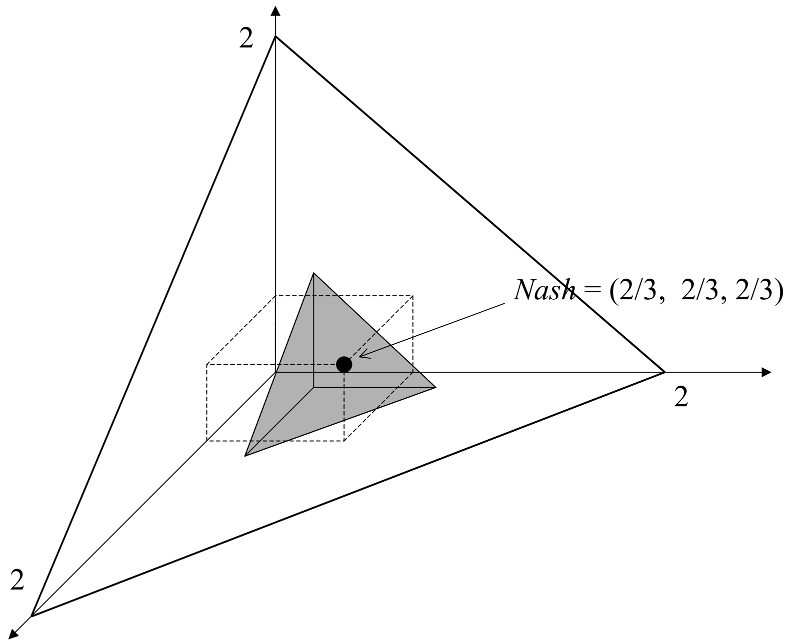

Figure 4

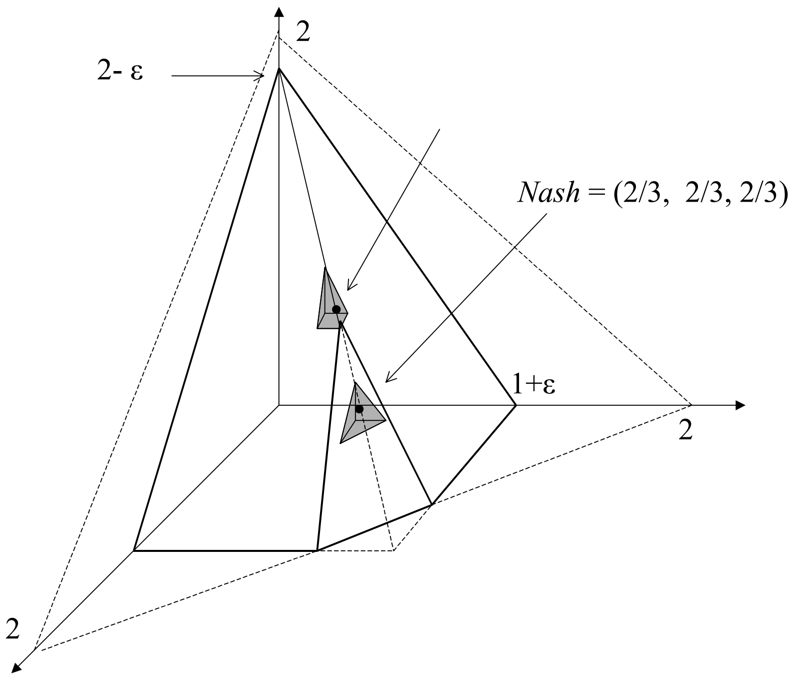

Figure 5.